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A link between quantum entanglement, secant varieties and sphericity

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Abstract

In this paper, we shed light on the relations between three concepts studied in representation theory, algebraic geometry and quantum information theory. First—spherical actions of reductive groups on projective spaces. Second—secant varieties of homogeneous projective varieties, and the related notions of rank and border rank. Third—quantum entanglement. Our main result concerns the relation between the problem of the state reconstruction from its reduced one-particle density matrices and the minimal number of separable summands in its decomposition. More precisely, we show that sphericity implies that states of a given rank cannot be approximated by states of a lower rank. We call states for which such an approximation is possible *exceptional states*. For three, important from a quantum entanglement perspective, cases of distinguishable, fermionic and bosonic particles, we also show that non-sphericity implies the existence of exceptional states. Remarkably, the exceptional states belong to non-bipartite entanglement classes. In particular, we show that the W -type states and their appropriate modifications are exceptional states stemming from the second secant variety for three cases above. We point out that the existence of the exceptional states is a physical obstruction for deciding the local unitary equivalence of states by means of the one-particle-reduced density matrices. Finally, for a number of systems of distinguishable particles with a known orbit structure, we list all exceptional states and discuss their possible importance in entanglement theory.

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1. Introduction

We consider a projective algebraic variety given as $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$, where \mathcal{H} is the state space of a composite system and \mathbb{X} consists of the coherent states of the system. The systems we consider are as follows:

- (i) L distinguishable particles with the state space $\mathcal{H}_D = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L$.
- (ii) L bosons (symmetric indistinguishable particles) with the state space $\mathcal{H}_B = S^L(\mathcal{H}_1)$, where \mathcal{H}_1 is the one-boson space.
- (iii) L fermions (antisymmetric indistinguishable particles) with the state space $\mathcal{H}_F = \bigwedge^L \mathcal{H}_1$, where \mathcal{H}_1 is the one-fermion space.

Throughout this paper, states will be understood as points in the projective space rather than vectors in the Hilbert space. Physically, it means that we neglect the norm and global phase of vectors. The respective varieties of coherent states, compact symmetry groups and their complexifications, which are local unitary (LU) and invertible SLOCC operations, respectively, are as follows:

- (i) The Segre embedding of $\mathbb{P}(\mathcal{H}_1) \times \cdots \times \mathbb{P}(\mathcal{H}_L)$ into $\mathbb{P}(\mathcal{H}_D)$, $K_D = SU(N_1) \times \cdots \times SU(N_L)$, $G_D = K_D^\mathbb{C} = SL(\mathcal{H}_1) \times \cdots \times SL(\mathcal{H}_L)$.
- (ii) The Veronese embedding of $\mathbb{P}(\mathcal{H}_1)$ into $\mathbb{P}(\mathcal{H}_B)$, $K = SU(N)$, $G = K^\mathbb{C} = SL(\mathcal{H}_1)$.
- (iii) The Plücker embedding of the Grassmannian $\text{Gr}(L, \mathcal{H}_1)$ into $\mathbb{P}(\mathcal{H}_F)$, $K = SU(N)$, $G = K^\mathbb{C} = SL(\mathcal{H}_1)$.

The LU operations represent the unitary operations which are performed on each particle separately, i.e. the LU dynamics. The SLOCC operations are more general and in addition allow classical communication. In all these cases, \mathbb{X} spans $\mathbb{P}(\mathcal{H})$, i.e. every state can be written as a linear combination of states from \mathbb{X} . The *rank* of a state is then defined as the minimal number of summands in such a linear combination. An obvious but important property is that the rank is invariant under the symmetries of the system.

Throughout this paper, when convenient, we will use Dirac bra–ket notation. The basis of \mathbb{C}^N will be denoted by $\{|0\rangle, \dots, |N-1\rangle\}$. The tensor $|i\rangle \otimes \cdots \otimes |j\rangle$ will typically be written as $|i \dots j\rangle$.

The notion of rank and the related secant varieties, which are introduced here in section 3, have been studied for a long time and from various perspectives—algebraic geometry, representation theory, computational complexity, algebraic statistics, etc; see e.g. [1, 7, 9, 20] and references therein. Much is known and much is still a mystery. We intend to show that these notions are also relevant in the study of quantum entanglement from the physics side and spherical varieties from the mathematical side. The first connection between quantum entanglement and secant varieties is rather recent and can be found in [15]. It is, however, conceptually different from what we present in the current paper.

The phenomenon which we focus on is the following. In some cases, states of a given rank can be approximated by states of lower rank. We will call such states *exceptional*, as to distinguish them from the prototypical case: the matrices, where the set of matrices whose rank does not exceed a given r is a closed set, so the approximation to a higher rank is impossible. We show that exceptional states occur for most systems of the above-mentioned types. In fact, they do not occur if and only if $L = 2$. Furthermore, the spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$, $S^2(\mathcal{H}_1)$, $\bigwedge^2 \mathcal{H}_1$ are exactly the ones where the complex symmetry group G acts spherically on $\mathbb{P}(\mathcal{H})$. In fact, for the tensor representations we are interested in, we observe that the absence of exceptional states is equivalent to sphericity of the group action. The central result is formulated as theorem 6.1. Theorem 6.1 provides a new definition of sphericity for the considered systems. Thus, the rather non-physical notion of a dense Borel group orbit is changed to a state-type obstruction.

Remarkably, the exceptional states we find, in all three cases, belong to non-bipartite SLOCC entanglement classes. More precisely, we show that W -type states are exceptional. For example, for the system of three qubits, $W = \frac{1}{3}(|001\rangle + |010\rangle + |001\rangle)$ is exceptional. Combining the results of the current paper with the recently established connection between sphericity and LU equivalence of states [17], we conclude in theorem 6.3 that the existence of the exceptional states can be regarded as a physical, state-type obstruction for deciding the LU equivalence of states by means of the one-particle-reduced density matrices.

From the point of view of representation theory, it is natural to ask whether the equivalence between the sphericity and the absence of exceptional states persists in a more general situation. The spherical actions of reductive groups on projective spaces are completely classified; see [19]. A case-by-case analysis using the list of [19] shows that sphericity implies the absence of exceptional states. This implication can also be deduced from a result of [7], where it is shown that rank and border rank coincide for a class of homogeneous projective varieties (subcominuscule varieties); the latter class is exactly the class obtained from spherical representations, taking account of the fact that sometimes several spherical representations give rise to the same variety of coherent states. However, the converse implication does not hold in general, i.e. there are irreducible representations of reductive groups, where exceptional states do not occur, and the action on the projective space is not spherical. Perhaps, the simplest example is given by the adjoint representation of $SL_3\mathbb{C}$. The conclusion is that we should regard the equivalence between sphericity and lack of exceptional states as a phenomenon present in the setting of quantum entanglement, but not beyond.

The fact that exceptional states do occur for systems with more than two particles can be qualified, from the point of view of algebraic geometry, as folklore. All necessary results can be found in [20]. The value of this work, as we see it, is not so much in the proofs, but rather in exhibiting a connection between three separately well-studied notions: rank, entanglement types and sphericity. Such a connection could, hopefully, lead to interactions and new developments. In particular, one could seek a relation between our work and the recent work [16], where some concepts of algebraic geometry, closely related to the ones used here, have recently been applied to the study of quantum entanglement (see also [6]).

This paper is organized as follows. We start with two simple motivating examples. The first describes a situation where exceptional states are absent and the second where they are present. Next, in section 3, we give rigorous definitions of rank, border rank and secant varieties and discuss some of their basic properties. In section 5, we present some known results about secant varieties for distinguishable particles, bosons and fermions. Sections 6 and 7 consist of the formulation and the proof of the main results of this paper. In section 8, we give a list of all exceptional states which appear for systems with the explicitly known orbit structure.

2. Motivation—a simple example

Before we proceed with the general definitions and results, we present two examples: one where exceptional states do not occur—the case of two qubits—and one where they do occur—three qubits. These systems are well studied from many aspects. In the theory of quantum entanglement, the exceptional state for three qubits is the so-called W -state.

Two qubits. Consider two complex vector spaces of dimension 2, $\mathcal{H}_1 \cong \mathcal{H}_2 \cong \mathbb{C}^2$. The tensor product $\mathcal{H}_D = \mathcal{H}_1 \otimes \mathcal{H}_2$, or more precisely, its projective space $\mathbb{P} = \mathbb{P}(\mathcal{H}_D)$ is the state space for a system of two qubits. This state space carries an action of the compact

group $K_D = SU(2) \times SU(2)$ and also of its complexification $G_D = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$. We concentrate on the action of G_D .

There are exactly two G_D -orbits in \mathbb{P} . The first consists of all simple tensors (separable states), i.e. tensors which can be written⁴ as $[v \otimes v']$ with $v \in \mathcal{H}_1$, $v' \in \mathcal{H}_2$. The second orbit consists of all non-simple tensors. It turns out that any state $[\psi]$ in the second orbit can be written as a sum of two simple tensors. More precisely, there exist bases v_1, v_2 of \mathcal{H}_1 and v'_1, v'_2 of \mathcal{H}_2 so that $\psi = v_1 \otimes v'_1 + v_2 \otimes v'_2$. Let us denote the first orbit by \mathbb{X} , so that the second one is $\mathbb{P} \setminus \mathbb{X}$. The set \mathbb{X} is actually a well-known algebraic variety, called the Segre embedding⁵ of $\mathbb{P}^1 \times \mathbb{P}^1$.

The basic notion considered in this paper is the notion of *rank* of states. In this terminology, the states of the first orbit will be said to have rank 1 and the states of the second orbit to have rank 2, according to the minimal number of simple tensors necessary to express a given state. This terminology has a classical origin: the space $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be interpreted as the space of 2×2 matrices with complex entries and we recover the standard notion of rank of a matrix.

Three qubits. Let $\mathcal{H}_D = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, with $\mathcal{H}_j \cong \mathbb{C}^2$. Let $\mathbb{P} = \mathbb{P}(\mathcal{H}_D)$. Let $G = SL(\mathcal{H}_1) \times SL(\mathcal{H}_2) \times SL(\mathcal{H}_3)$ act naturally on \mathbb{P} . Let $\mathbb{X} \subset \mathbb{P}$ denote the variety of simple tensors; this is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into \mathbb{P} . Let e_1, e_2 be a basis of \mathbb{C}^2 . Then, the orbits of G in \mathbb{P} are the following (cf [14]).

$$\begin{aligned} \mathbb{X} &= \mathbb{X}_1 = G[e_1 \otimes e_1 \otimes e_1] \\ \mathbb{X}_2^{\text{I}} &= G[e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2] \\ \mathbb{X}_2^{\text{II}} &= G[e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_2] \\ \mathbb{X}_2^{\text{III}} &= G[e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_1 \otimes e_2] \\ \mathbb{X}_2^{\text{IV}} &= G[e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_1] \\ \mathbb{X}_3 &= G[e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1]. \end{aligned}$$

The lower index indicates rank. The upper index for \mathbb{X}_2 distinguishes the four different orbits of rank 2; observe that cases II–IV differ only by permutation of the indices; these are so-called bi-separable states. It is not hard to see that the orbit \mathbb{X}_2^{I} is open and dense in \mathbb{P} . Hence, all states can be approximated by states from \mathbb{X}_2^{I} . In particular, this is true for the state

$$W = [e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1].$$

We say that W has border rank 2. However, it can be checked directly that W cannot be written as a sum of two simple tensors, so its rank is equal to 3. Thus, W is an exceptional state—it can be approximated by states of lower rank. At the same time, the states $e_1 \otimes e_1 \otimes e_1 + e_2 \otimes e_2 \otimes e_2$ and W are known to represent the two genuine entanglement classes of three qubits.

3. Rank of states and secant varieties: general definitions

After the simple motivating examples given above, we proceed with the rigorous definitions of the central notions of this paper: rank, border rank and secant varieties. We also state some of their basic properties. A detailed treatment can be found in [20], along with an account of the state of affairs for the cases concerning us here, from the point of view of representation theory and algebraic geometry.

Let \mathcal{H} denote a complex vector space of dimension N and $\mathbb{P} = \mathbb{P}(\mathcal{H})$ denote the associated projective space. If $v \in \mathcal{H}$ is a nonzero vector, we shall denote by $[v]$ its image in \mathbb{P} .

⁴ We denote by $[\psi]$ the point in $\mathbb{P}(\mathcal{H})$ corresponding to a vector $\psi \in \mathcal{H} \setminus \{0\}$.

⁵ Incidentally, \mathbb{X} is also a quadric hypersurface, but we regard it here as a Segre embedding since this is what we want to consider in a more general situation.

Rank and maximal rank. Let $\mathbb{X} \subset \mathbb{P}$ be an algebraic variety and $\hat{\mathbb{X}} \subset \mathcal{H}$ denote the affine cone over \mathbb{X} . Then, $\text{span}\hat{\mathbb{X}}$ is a linear subspace of \mathcal{H} . We denote by $\mathbb{P}_{\mathbb{X}} = \mathbb{P}(\text{span}\hat{\mathbb{X}})$ the corresponding projective subspace of \mathbb{P} . We say that \mathbb{X} spans \mathbb{P} if $\mathbb{P}_{\mathbb{X}} = \mathbb{P}$; this is equivalent to the requirement that $\hat{\mathbb{X}}$ contains a basis of \mathcal{H} . Assume that this is the case. Then, every point in \mathcal{H} can be written as a linear combination of points in $\hat{\mathbb{X}}$. This allows us to define the following notion of rank of a vector with respect to \mathbb{X} , for nonzero $\psi \in \mathcal{H}$:

$$\text{rk}[\psi] = \text{rk}_{\mathbb{X}}[\psi] = \min\{r \in \mathbb{N} : \psi = x_1 + \cdots + x_r \text{ with } [x_j] \in \mathbb{X}\}. \quad (1)$$

The sets

$$\mathbb{X}_r = \{[\psi] \in \mathbb{P} : \text{rk}[\psi] = r\}, \quad r = 1, 2, \dots, \quad (2)$$

are called the rank subsets of \mathbb{P} with respect to \mathbb{X} . Since \mathbb{X} spans \mathbb{P} , we have $\mathbb{X}_r = \emptyset$ for $r > N$. In the following proposition, we state some basic properties of the rank subsets. We skip the formal proof since all properties follow immediately from the definitions.

Proposition 3.1. *The following properties hold:*

- (i) $\mathbb{X}_1 = \mathbb{X}$.
- (ii) There exists $r_m \in \{1, \dots, N\}$, such that $\mathbb{X}_{r_m} \neq \emptyset$ and $\mathbb{X}_r = \emptyset$ for $r > r_m$.
The number r_m is called the maximal rank of \mathbb{P} with respect to \mathbb{X} .
- (iii) If $r \in \{1, \dots, r_m\}$, then $\mathbb{X}_r \neq \emptyset$.
- (iv) The projective space \mathbb{P} decomposes as a disjoint union $\mathbb{P} = \mathbb{X}_1 \sqcup \dots \sqcup \mathbb{X}_{r_m}$.

Secant varieties, typical rank. Let $r \in \{2, \dots, r_m\}$ and $\mathbb{X}_r \subset \mathbb{P}$ be as defined in (2). Let us first note that the subset $\mathbb{X}_r \subset \mathbb{P}$ is not closed. It can be easily seen because we have $\mathbb{X} \subset \overline{\mathbb{X}_r}$ and $\mathbb{X} \not\subset \mathbb{X}_r$.⁶ The r th secant variety of \mathbb{X} is defined as

$$\sigma_r(\mathbb{X}) = \overline{\bigcup_{x_1, \dots, x_r \in \mathbb{X}} \mathbb{P}_{x_1, \dots, x_r}}, \quad (3)$$

where $\mathbb{P}_{x_1, \dots, x_r}$ stands for the projective subspace of \mathbb{P} spanned by the points x_1, \dots, x_r . It can also be written as

$$\sigma_r(\mathbb{X}) = \overline{\bigsqcup_{s \leq r} \mathbb{X}_s} \subset \mathbb{P}. \quad (4)$$

In the proposition below, we state some, easy to verify, basic properties of secant varieties.

Proposition 3.2. *The following properties hold:*

- (i) $\sigma_1(\mathbb{X}) = \mathbb{X}_1 = \mathbb{X}$.
- (ii) $\sigma_r(\mathbb{X}) \subset \sigma_{r+1}(\mathbb{X})$.
- (iii) There exists a minimal $r_g \in \{1, \dots, r_m\}$ such that $\sigma_{r_g}(\mathbb{X}) = \mathbb{P}$ and $\sigma_{r_g-1}(\mathbb{X}) \neq \mathbb{P}$.
This number is called the typical rank of \mathbb{P} with respect to \mathbb{X} .
- (iv) For $r \in \{1, \dots, r_g\}$, the rank subset \mathbb{X}_r is Zariski open and dense in $\sigma_r(\mathbb{X})$ and we have $\sigma_r(\mathbb{X}) = \overline{\mathbb{X}_r}$.

⁶ Here and in what follows, we use \overline{S} to denote the Zariski closure of a subset $S \subset \mathbb{P}$.

Border rank and exceptional states. As we have already seen for three qubits, it may happen that a state with a given rank can be approximated by states of strictly lower rank. Here, we introduce the notion of a border rank which captures this kind of behavior. Let $[\psi] \in \mathbb{P}$. The border rank of $[\psi]$ with respect to \mathbb{X} is defined as

$$\underline{\text{rk}}[\psi] = \underline{\text{rk}}_{\mathbb{X}}[\psi] = \min\{r \in \mathbb{N} : [\psi] \in \overline{\mathbb{X}}_r\}.$$

Proposition 3.3. *The following properties hold:*

- (i) $\text{rk}[\psi] \geq \underline{\text{rk}}[\psi]$.
- (ii) $\underline{\text{rk}}[\psi] = \min\{r \in \mathbb{N} : [\psi] \in \sigma_r(\mathbb{X})\}$.

Definition 3.1. *States $[\psi] \in \mathbb{P}$, for which $\text{rk}[\psi] \neq \underline{\text{rk}}[\psi]$, are called exceptional.*

Clearly, if $[\psi]$ is exceptional, then $\underline{\text{rk}}[\psi] < \text{rk}[\psi]$, so exceptional states are states which can be approximated by states of lower rank.

Expected dimensions of secant varieties. We can easily give a set of parameters sufficient to determine a point on $\sigma_r(\mathbb{X})$; this gives an upper bound on the dimension of the secant variety. Namely a generic point in $\sigma_r(\mathbb{X})$ is obtained as follows. We take r points on \mathbb{X} , which give $r \dim \mathbb{X}$ parameters, and then, assuming that these points are linearly independent, we take a point in the $(r - 1)$ -dimensional projective space spanned on them. In total, we obtain $r \dim \mathbb{X} + (r - 1)$ parameters. Since there are no obvious relations between these parameters, this number is called the *expected dimension* of $\sigma_r(\mathbb{X})$, when it does not exceed $N - 1 = \dim \mathbb{P}$. It is denoted by

$$\text{e dim } \sigma_r(\mathbb{X}) = \min\{r \dim \mathbb{X} + (r - 1), N - 1\}.$$

If $\dim \sigma_r(\mathbb{X}) \neq \text{e dim } \sigma_r(\mathbb{X})$, the secant variety is called defective and the difference between the two dimensions is called the *defect*.

With this notion in hand, we can calculate an expected value r_{eg} for the typical rank of \mathbb{P} with respect to \mathbb{X} , called the expected typical rank. This is the minimal r for which $r \dim \mathbb{X} + (r - 1) \geq N - 1$, so

$$r_{\text{eg}} = \left\lceil \frac{N}{\dim \mathbb{X} + 1} \right\rceil.$$

Clearly, $r_g \leq r_{\text{eg}}$.

4. Distinguishable particles, bosons and fermions

In the following, we show that the algebraic varieties of coherent states, for our three cases of interest, in fact span the respective projective spaces \mathbb{P} . This in turn implies that for distinguishable particles, bosons and fermions, there are well-defined notions of rank and border rank.

Distinguishable particles and the Segre variety. Let $\mathcal{H}_1, \dots, \mathcal{H}_L$ be complex vector spaces with finite dimensions $n_j = \dim \mathcal{H}_j$. Let $G_D = GL(\mathcal{H}_1) \times \dots \times GL(\mathcal{H}_L)$. Let $\mathcal{H}_D = \mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_L$ and $\mathbb{P} = \mathbb{P}(\mathcal{H}_D)$. Then, $\dim \mathcal{H}_D = N = n_1 \times \dots \times n_L$ and $\dim \mathbb{P} = N - 1$. The group G_D acts naturally on \mathcal{H}_D and this representation is irreducible. There is of course an induced action on \mathbb{P} . Consider the map

$$\begin{aligned} \text{Segre} : \mathbb{P}(\mathcal{H}_1) \times \dots \times \mathbb{P}(\mathcal{H}_L) &\longrightarrow \mathbb{P} \\ ([v_1], \dots, [v_L]) &\longmapsto [v_1 \otimes \dots \otimes v_L]. \end{aligned}$$

It is not hard to see that this map defines a G_D -equivariant embedding, called the Segre embedding of $\mathbb{P}(\mathcal{H}_1) \times \cdots \times \mathbb{P}(\mathcal{H}_L)$. We denote the image of the Segre map by $\mathbb{X} \subset \mathbb{P}$. This is an algebraic variety, called the Segre variety; it consists of (the projectivizations of) all simple tensors. The Segre variety is the unique closed G_D -orbit in \mathbb{P} . In physical terms, \mathbb{X} is the orbit of coherent or separable states.

The variety \mathbb{X} spans \mathbb{P} . Indeed, if we fix bases for $\mathcal{H}_1, \dots, \mathcal{H}_L$, then the tensor products of the basis vectors form a basis of simple tensors for \mathcal{H}_D . Thus, every tensor can be expressed as a linear combination of simple tensors and we have well-defined notions of rank and border rank of states with respect to \mathbb{X} , as defined in section 3.

Bosons and the Veronese variety. Let \mathcal{H}_1 be a complex vector space of dimension n . Let $G = GL(\mathcal{H}_1)$. Let $\mathcal{H}_B = S^L(\mathcal{H}_1)$, for some fixed positive integer L and $\mathbb{P} = \mathbb{P}(\mathcal{H}_B)$. Then, $\dim \mathcal{H}_B = N = \binom{L+n-1}{L}$ and $\dim \mathbb{P} = N - 1$. The group G acts naturally on \mathcal{H}_B and this representation is irreducible. There is an induced action on \mathbb{P} . Consider the map

$$\begin{aligned} \text{Ver}_L : \mathbb{P}(\mathcal{H}_1) &\longrightarrow \mathbb{P} \\ [v] &\longmapsto [v^L]. \end{aligned}$$

This map is an embedding, called the L th Veronese embedding of $\mathbb{P}(\mathcal{H}_1)$. Let us denote the image by $\mathbb{X} \subset \mathbb{P}$. This is an algebraic subvariety of \mathbb{P} called the Veronese variety. It consists of all symmetric L -tensors over \mathcal{H}_1 which are powers of 1-tensors. Since G acts transitively on $\mathbb{P}(\mathcal{H}_1)$ and the map Ver_L is G -equivariant, \mathbb{X} is a single closed G -orbit. Since the representation of G on \mathcal{H}_B is irreducible, \mathbb{X} is the only closed G -orbit in \mathbb{P} .

The variety \mathbb{X} spans \mathbb{P} . This is, perhaps, not as obvious as in the case of the Segre variety because the standard basis for $S^L(\mathcal{H}_1)$ is the monomial basis obtained from a given basis $\{e_1, \dots, e_n\}$ of \mathcal{H}_1 . The only monomials which belong to \mathbb{X} are $[e_j^L]$. The other monomials, like $[e_1 e_2^{L-1}]$, do not belong to \mathbb{X} . To see that \mathbb{X} actually spans \mathbb{P} , observe that, since \mathbb{X} is a closed G -orbit, $\text{span} \mathbb{X}$ gives a subspace of \mathcal{H}_B preserved by G , i.e. a subrepresentation. But \mathcal{H} is irreducible, so we must have $\text{span} \mathbb{X} = \mathbb{P}$. Hence, we have well-defined notions of rank and border rank in \mathbb{P} with respect to \mathbb{X} .

Fermions and the Grassmann variety. Let \mathcal{H}_1 be a complex vector space of dimension n and $G = GL(\mathcal{H}_1)$. Let $\text{Gr}(L, \mathcal{H}_1)$ denote the L th Grassmann variety, for some fixed integer L with $1 \leq L \leq n$; this is the variety of all L -dimensional linear subspaces of \mathcal{H}_1 . Then, $\text{Gr}(L, \mathcal{H}_1)$ carries a natural G -action. Let $\mathcal{H}_F = \bigwedge^L \mathcal{H}_1$ and $\mathbb{P} = \mathbb{P}(\mathcal{H}_F)$. Then, $\dim \mathcal{H}_F = N = \binom{n}{L}$ and $\dim \mathbb{P} = N - 1$. There is a natural linear representation of G on \mathcal{H}_F , which is irreducible. Consider the map

$$\begin{aligned} \text{Pl}_L : \text{Gr}(L, \mathcal{H}_1) &\longrightarrow \mathbb{P} \\ U &\longmapsto [u_1 \wedge \cdots \wedge u_L], \quad \text{where } u_1, \dots, u_L \text{ is a basis of } U. \end{aligned}$$

Here, $U \subset \mathcal{H}_1$ is an L -dimensional subspace, i.e. an element of $\text{Gr}(L, \mathcal{H}_1)$. The image $\text{Pl}_L(U)$ is well defined, i.e. does not depend on the choice of basis because for two different bases $\{u_j\}$ and $\{u'_j\}$, the exterior products $u_1 \wedge \cdots \wedge u_L$ and $u'_1 \wedge \cdots \wedge u'_L$ differ only by a scalar, so in the projective space \mathbb{P} , we have $[u_1 \wedge \cdots \wedge u_L] = [u'_1 \wedge \cdots \wedge u'_L]$. The map Pl_L is a G -equivariant embedding, called the Plücker embedding of the Grassmann variety. Let us denote the image by $\mathbb{X} = \text{Pl}_L(\text{Gr}(L, \mathcal{H}_1))$. Then, \mathbb{X} is the unique closed G -orbit in \mathbb{P} ; it consists of all decomposable antisymmetric tensors in \mathcal{H}_F . The variety \mathbb{X} spans \mathbb{P} because it contains the standard basis of \mathcal{H}_F consisting of the exterior products of L -tuples of elements of a given basis of \mathcal{H}_1 . Hence, we have well-defined notions of rank and border rank in \mathbb{P} with respect to \mathbb{X} .

Remark 4.1. Unless otherwise specified, we shall only consider the cases where $L \leq n/2$. This is sufficient for our purposes because of an isomorphism $\bigwedge^L \mathcal{H}_1 \cong \bigwedge^{n-L} \mathcal{H}_1$. Such an isomorphism is not canonical; in fact, the two spaces are dual to each other. So, we can obtain an isomorphism by choosing a positive-definite Hermitian form on \mathcal{H}_1 . Then, the mapping $\text{Gr}(L, \mathcal{H}_1) \rightarrow \text{Gr}(L, \mathcal{H}_1)$, given by $U \mapsto U^\perp$, is an isomorphism which agrees with the respective Plücker embeddings. The Hermitian form cannot be G -equivariant, but the associated unitary group $K = U(\mathcal{H}_1)$ is a maximal compact subgroup of G . Thus, although the isomorphism $\bigwedge^L \mathcal{H}_1 \cong \bigwedge^{n-L} \mathcal{H}_1$ is not G -equivariant, there is a 1-1 correspondence between the G -orbits. Also, since the isomorphism sends simple tensors to simple tensors, the rank and border rank behave in the same way.

5. A short overview of known results about secant varieties

The literature on secant varieties is vast and diverse, and a real overview is a task we do not intend to pursue here. We only give a very small sample of theorems concerning the situations described in section 4. Although the only result we will need is theorem 5.4, we found that it is useful to state some known results at least for the most interesting cases of quantum entanglement. The interested reader is referred to [20] for an extensive overview of the subject.

L-qubit system. Assume $n_j = 2$, i.e. $\mathcal{H}_j = \mathbb{C}^2$, for all j . Thus, $\mathcal{H}_D = (\mathbb{C}^2)^{\otimes L}$ and the Segre variety \mathbb{X} is an embedding of the L -fold product $\mathbb{P}^1 \times \dots \times \mathbb{P}^1$ into \mathbb{P}^{2^L-1} . The dimensions of all secant varieties for this case have been determined in [11]. Below, we state the main results from this paper.

Theorem 5.1 ([11]). *The dimension of the secant variety $\sigma_r(\mathbb{X})$ equals the expected dimension $rL + r - 1$ in all cases but one. The exceptional case is $r = 3, L = 4$, i.e. four qubits and σ_3 , where $\dim \sigma_3(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 13$, while the expected dimension is 14.*

Corollary 5.2. *The minimal number of simple tensors in $(\mathbb{C}^2)^{\otimes L}$ necessary to express a generic tensor as a linear combination is the expected number $r_g = r_{\text{eg}} = \left\lceil \frac{2^L}{L+1} \right\rceil$.*

L-boson system. The following theorem of Alexander and Hirschowitz determines which secant variety to the Veronese variety fills the ambient projective space.

Theorem 5.3 ([5]). *A generic element of $S^L(\mathcal{H}_1)$ with $\mathcal{H}_1 \cong \mathbb{C}^n$, i.e. a generic state of L n -level bosons, can be written as the sum of the expected number $r_{\text{eg}} = \left\lceil \frac{1}{n} \binom{n-1+L}{L} \right\rceil$, and no fewer, L th powers of elements of \mathcal{H}_1 , with the following exceptions: $(n, L) = (3, 4), (4, 4), (5, 4), (5, 3)$ and $(n, 2)$ for all $n \geq 2$.*

The problem of determining the rank of a symmetric tensor with respect to the Veronese variety is known as Waring's problem. A result which plays a role in the next section concerns the rank of a monomial. Below, we state two theorems of Carlini, Catalisano and Geramita, solving Waring's problem for monomials and sums of coprime monomials. Let e_1, \dots, e_n be a basis of \mathcal{H}_1 . A monomial in $S^k(\mathcal{H}_1)$ is a product of powers of the basis elements, i.e. $M = e_1^{\alpha_1}, \dots, e_n^{\alpha_n}$ with $\alpha_1 + \dots + \alpha_n = L$. The basis can be reordered so that the nonzero exponents are increasing and the zero exponents are at the end, i.e. $M = e_1^{\alpha_1}, \dots, e_m^{\alpha_m}$ with $1 \leq \alpha_1 \leq \dots \leq \alpha_m$ and $m \leq n$.

Theorem 5.4 ([8]). *Let $M \in S^L(\mathcal{H}_1)$ be a monomial as above. Then,*

$$\text{rk}(M) = \prod_{j=2}^m (\alpha_j + 1).$$

Examples show that the rank of a monomial may be greater or smaller than the rank r_g of a generic element of $S^L(\mathcal{H}_1)$.

Theorem 5.5 ([8]). *Suppose $f \in S^L(\mathcal{H}_1)$ has the form $f = M_1 + \dots + M_l$, where M_1, \dots, M_l are pairwise coprime monomials of degree L . Then,*

$$\text{rk}(f) = \sum_{j=1}^l \text{rk}(M_j).$$

Furthermore, if $M = M_1 \dots M_l$ denotes the product, then

$$\text{rk}(M_j) \leq \text{rk}(f) \leq \text{rk}(M).$$

L-fermion system. Results about the secant varieties of Grassmann varieties were obtained in [10]. In particular, the following theorem is proven therein.

Theorem 5.6 ([10]). *Let $\mathbb{X} = \text{Pl}(\text{Gr}(L, \mathbb{C}^n)) \subset \mathbb{P}(\bigwedge^L \mathbb{C}^n)$ be the image of the Plücker embedding of the Grassmann variety, with $L \leq n/2$ (see section 4 for details).*

- (i) *If $L = 2$, then the typical rank is $r_g = \lfloor \frac{n}{2} \rfloor$. For $1 < r < r_g$, the secant variety $\sigma_r(\mathbb{X})$ is defective with defect $e \dim \sigma_r(\mathbb{X}) - \dim \sigma_r(\mathbb{X}) = 2r(s-1)$.*
- (ii) *If $L \geq 3$ and $Lr \leq n$, then $\sigma_r(\mathbb{X})$ has the expected dimension $r \dim \mathbb{X} + r - 1$.*

6. The main results

Let G be a reductive complex Lie group, i.e. G equals the complexification $K^{\mathbb{C}}$ of a compact Lie group K . Recall that a Borel subgroup $B \subset G$ is a maximal solvable subgroup and all Borel subgroups of G are conjugate. In the case when G is $SL_N(\mathbb{C})$, an example of a Borel subgroup is given by all upper triangular matrices in G . In the case G splits as a direct product $G = G_1 \times \dots \times G_L$, any Borel subgroup of G is obtained as a product $B = B_1 \times \dots \times B_k$, where B_j is a Borel subgroup of G_j .

Suppose that G acts on an algebraic (affine or projective) variety \mathbb{Y} . We say that \mathbb{Y} is a G -variety. The variety \mathbb{Y} is called a spherical G -variety, if a Borel subgroup B of G has an open dense orbit in \mathbb{Y} .

Let \mathbb{Y} be a spherical G -variety. Since some, and therefore any, Borel subgroup B of G has an open dense orbit in \mathbb{Y} , it follows that G also has an open dense orbit in \mathbb{Y} . Let $y \in \mathbb{Y}$ be a point from this open dense G -orbit and G_y be the isotropy group at this point. Then, the open G -orbit in \mathbb{Y} is isomorphic to the coset space G/G_y . The open B -orbit is necessarily contained in the open G -orbit. Thus, B has an open orbit in G/G_y .

We are mostly concerned with a very specific type of spherical varieties described as follows. Let $\rho : G \rightarrow GL(\mathcal{H})$ be a linear representation of a reductive group G . If the resulting action of G on \mathcal{H} is spherical, we say that ρ is a *spherical representation*. In such a case, it is easy to see that the induced action of G on the projective space $\mathbb{P} = \mathbb{P}(\mathcal{H})$ is also spherical. The converse is not automatically true, but the problem can easily be removed as follows. Suppose G acts spherically on $\mathbb{P}(\mathcal{H})$ via a linear representation ρ as above. Let $\tilde{G} = \mathbb{C}^\times \times G$ and consider the representation

$$\tilde{\rho} : \tilde{G} \longrightarrow GL(\mathcal{H}), \quad \rho(\lambda, g) = \lambda \rho(g).$$

This representation defines a spherical action of \tilde{G} on \mathcal{H} .

Suppose now $\rho : G \longrightarrow GL(\mathcal{H})$ is a spherical representation which is in addition irreducible. Then, G has a unique closed orbit $\mathbb{X} \subset \mathbb{P}$. We may consider the secant varieties $\sigma_r(\mathbb{X})$ of \mathbb{X} in \mathbb{P} . The irreducibility implies that \mathbb{X} spans \mathbb{P} . Thus, we have well-defined rank and border rank functions on \mathbb{P} with respect to \mathbb{X} . The main theorem of this paper, which we prove in the next section, is the following.

Theorem 6.1. *Suppose that we have one of the following three configurations⁷ of a state space \mathcal{H} , a complex reductive Lie group G acting irreducibly on \mathcal{H} and a variety of coherent states $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$, which is the unique closed G -orbit in the projective space $\mathbb{P}(\mathcal{H})$.*

- (i) $\mathcal{H}_D = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L$, $G_D = GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_L)$, $\mathbb{X} = \text{Segre}(\mathbb{P}(\mathcal{H}_1) \times \cdots \times \mathbb{P}(\mathcal{H}_L))$.
- (ii) $\mathcal{H}_B = S^L(\mathcal{H}_1)$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \text{Ver}_L(\mathbb{P}(\mathcal{H}_1))$.
- (iii) $\mathcal{H}_F = \bigwedge^L \mathcal{H}_1$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \text{Pl}(\text{Gr}(L, \mathcal{H}_1))$.

Then, the action of G on $\mathbb{P}(\mathcal{H}_{B,F})$ (resp. G_D on $\mathbb{P}(\mathcal{H}_D)$) is spherical if and only if there do not exist exceptional states in $\mathbb{P}(\mathcal{H}_{B,F})$ (resp. $\mathbb{P}(\mathcal{H}_D)$) with respect to \mathbb{X} . In other words, sphericity of the representation is equivalent to the property that states of any given rank cannot be approximated by states of lower rank.

Note that theorem 6.1 provides a new definition of sphericity for the considered systems. The notion of rather non-physical dense Borel group orbit has been changed to a state-type obstruction. In the next section, we will see that some exceptional states belong to non-bipartite SLOCC entanglement classes. In particular, the W -type state is exceptional. Hence, the obstruction to sphericity is not only state-type but also entanglement-type.

Recently, the importance of spherical varieties for the problem of LU equivalence of states has been pointed out by the authors of [17]. Two states of L distinguishable particles, L bosons or L fermions are called LU equivalent if and only if they can be connected by the action of $K_D = SU(N_1) \times \cdots \times SU(N_L)$, $K = SU(N)$ or $K = SU(N)$, respectively. It is well known that complex projective spaces are Kähler and hence symplectic manifolds. The action of K_D on $\mathbb{P}(\mathcal{H}_D)$ and action of K on $\mathbb{P}(\mathcal{H}_{B,F})$ are Hamiltonian and therefore there exist the momentum maps $\mu : \mathbb{P}(\mathcal{H}_D) \rightarrow \mathfrak{k}_D^*$ and $\mu : \mathbb{P}(\mathcal{H}_{B,F}) \rightarrow \mathfrak{k}^*$. Using the customary identification of Lie algebra with its dual, one can show that [24]

$$\mu([v]) = \frac{i}{2} \{\tilde{\rho}_1([v]), \dots, \tilde{\rho}_L([v])\} \text{ for distinguishable particles,} \quad (5)$$

$$\mu([v]) = \frac{i}{2} \tilde{\rho}_1([v]) \text{ for bosons and fermions,} \quad (6)$$

where in all three cases, $\tilde{\rho}_i([v]) = \rho - \frac{1}{n_i} I_{n_i}$ and $\rho_i([v])$ are the one-particle-reduced density matrices of a state $[v]$. The natural question stemming from the LU equivalence problem is: when could one decide about LU equivalence by means of reduced density matrices only? The answer to this question is provided by Brion's theorem, [2], stating that an irreducible $K^\mathbb{C}$ -variety is spherical if and only if the momentum map distinguishes K -orbits. The formulation of this theorem adjusted to our setting reads as follows.

Theorem 6.2. *Let K be a connected compact Lie group acting on $\mathbb{P}(\mathcal{H})$ by a Hamiltonian action and let $G = K^\mathbb{C}$. The following are equivalent:*

- (i) $\mathbb{P}(\mathcal{H})$ is a spherical variety.
- (ii) Two states $[v_1]$ and $[v_2]$ are LU equivalent if and only if the spectra of their reduced one-particle density coincide.

⁷ See section 4 for details.

Combining theorem 6.1 with the above theorem, we obtain the following.

Theorem 6.3. *The existence of the exceptional states is a physical, state-type obstruction for deciding the LU equivalence by means of the one-particle-reduced density matrices.*

Remark 6.1. As we mentioned in section 2, for three qubits, the only exceptional states are W -type states which have rank 3 and border rank 2. The states of rank 2 which can approximate the state W belong to \mathbb{X}_2^1 , which is the SLOCC class of gigahertz state. Note that if we remove this class from $\mathbb{P}(\mathcal{H})$, the remaining states, i.e. the closure of the SLOCC class of W , are non-exceptional. On the other hand, in [26], it was shown that the closure of the SLOCC class of W is spherical, i.e. it is possible to decide the LU equivalence by means of the one-particle-reduced density matrices for W -type states. In the light of theorem 6.3, one can argue that it is because we removed from $\mathbb{P}(\mathcal{H})$ the states which were responsible for the appearance of exceptional states. The generalization of this kind of reasoning to other systems is not immediate; however, it might give an insight into the classification of exceptional states.

7. The proof

In this section, we present a proof of the main results stated in section 6. The proof is organized as follows. First, in theorem 7.1, we show that sphericity implies lack of exceptional states. Next, we proceed with a proof of the opposite implication for L distinguishable particles, L bosons and L fermions. It essentially consists of two parts. The first one reduces the problem to some particular low-dimensional cases (see section 7.1.1). Then, we prove the result for these cases, which finishes the proof of theorem 6.1.

Theorem 7.1. *Let $G \rightarrow GL(\mathcal{H})$ be an irreducible representation of a reductive complex Lie group G , such that the action of G on $\mathbb{P}(\mathcal{H})$ is spherical. Let $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$ be the closed G -orbit. Then, rank and border rank on $\mathbb{P}(\mathcal{H})$ with respect to \mathbb{X} coincide, i.e.*

$$\text{rk}_{\mathbb{X}}[\psi] = \text{rk}_{\mathbb{X}}[\psi],$$

for all $[\psi] \in \mathbb{P}$. In other words, there are no exceptional states in \mathbb{P} .

Proof. All spherical representations of reductive groups are classified; see [19]. The theorem can be proved case by case using Knop's list. In particular, for the systems of our interest, the sphericity is present only when $L = 2$ (see [17] for a detailed discussion). So we need to consider arbitrary 2-tensors for distinguishable particles and symmetric or antisymmetric 2-tensors for bosons and fermions. Note, however, that any such tensor ψ can be represented by a matrix M , which is arbitrary, symmetric or antisymmetric, respectively. We have $\text{rk}_{\mathbb{X}}[\phi] = \text{rk}M$ in the cases of distinguishable particles and symmetric tensors, and $\text{rk}_{\mathbb{X}}[\psi] = \frac{1}{2}\text{rk}M$ in the case of fermions. The set of matrices of rank smaller or equal to a given r is a closed set. Hence, the rank and the border rank coincide. \square

7.1. Absence of exceptional states implies sphericity

In this section, we prove the following result.

Theorem 7.2. *Suppose that we have one of the following three configurations⁸ of a state space \mathcal{H} , a complex reductive Lie group G acting irreducibly on \mathcal{H} and a variety of coherent states $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$, which is the unique closed G -orbit in the projective space $\mathbb{P}(\mathcal{H})$.*

⁸ See section 4 for details.

- (i) $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L$, $G = GL(\mathcal{H}_1) \times \cdots \times GL(\mathcal{H}_L)$, $\mathbb{X} = \text{Segre}(\mathbb{P}(\mathcal{H}_1) \times \cdots \times \mathbb{P}(\mathcal{H}_L))$, with $n_j = \dim V_j \geq 2$.
- (ii) $\mathcal{H} = S^L(\mathcal{H}_1)$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \text{Ver}_L(\mathbb{P}(\mathcal{H}_1))$, with $n = \dim \mathcal{H}_1 \geq 2$.
- (iii) $\mathcal{H} = \bigwedge^L \mathcal{H}_1$, $G = GL(\mathcal{H}_1)$, $\mathbb{X} = \text{Pl}_L(\text{Gr}(L, \mathcal{H}_1))$, with $n = \dim \mathcal{H}_1 \geq 6$.

Suppose that $L \geq 3$ and, only in case (iii), also $n - L \geq 3$. Then, there exist exceptional states in $\mathbb{P}(\mathcal{H})$. In other words, there exist states in $\mathbb{P}(\mathcal{H})$ which can be approximated by states of lower rank. Moreover, exceptional states already occur in the second secant variety $\sigma_2(\mathbb{X})$.

Remark 7.1. The assumption $L \geq 3$ and, in case (iii), $n - L \geq 3$, are equivalent to the assumption that the action of G on \mathcal{H} is not spherical. Thus, the above theorem can be phrased as: non-sphericity implies the presence of exceptional states. Combined with theorem 7.1, this implies that, for the cases of interest, sphericity is equivalent to the lack of exceptional states, thus proving theorem 6.1.

The strategy for the proof of theorem 7.2 is as follows. In sections 7.1.2–7.1.4, we find some basic examples of exceptional states in the smallest non-spherical representations. In section 7.1.1, we prove the reduction lemma which allows a reducing calculation of rank in the general case to these small-dimensional cases. Then at the end of section 7, we combine the above results and this way finish the proof of theorem 7.2.

7.1.1. Reduction to lower dimensional cases. In this section, we consider particular cases of a state space \mathcal{H} and a subspace $\tilde{\mathcal{H}} \subset \mathcal{H}$. There is a natural embedding $\mathbb{P}(\tilde{\mathcal{H}}) \subset \mathbb{P}(\mathcal{H})$. We have the variety of coherent states $\mathbb{X} \subset \mathbb{P}(\mathcal{H})$ for the first system. The variety of coherent states for the subsystem is of the form $\tilde{\mathbb{X}} = \mathbb{X} \cap \mathbb{P}(\tilde{\mathcal{H}})$. Now for any state $[\psi] \in \mathbb{P}(\tilde{\mathcal{H}})$, we have two notions of rank: with respect to \mathbb{X} or $\tilde{\mathbb{X}}$. We shall prove that, in our three cases of interest, these ranks agree, i.e. $\text{rk}_{\mathbb{X}}[\psi] = \text{rk}_{\tilde{\mathbb{X}}}[\psi]$. This will enable us to compute rank for some basic low-dimensional systems and deduce results for much more general situations. We start by defining explicitly the spaces \mathcal{H} and $\tilde{\mathcal{H}}$ we want to consider for distinguishable particles, bosons and fermions.

Case 1: Distinguishable particles. Let $\mathcal{H}_1, \dots, \mathcal{H}_L$ be finite-dimensional complex vector spaces and $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_L$. Let $U_j \subset \mathcal{H}_j$ be a subspace, for $j = 1, \dots, L$, and $\tilde{\mathcal{H}} = U_1 \otimes \cdots \otimes U_L$. Then, we have natural embeddings $\tilde{\mathcal{H}} \subset \mathcal{H}$ and $\mathbb{P}(\tilde{\mathcal{H}}) \subset \mathbb{P}(\mathcal{H})$. The respective varieties of coherent states are $\mathbb{X} = \text{Segre}(\mathbb{P}(\mathcal{H}_1) \times \cdots \times \mathbb{P}(\mathcal{H}_L)) \subset \mathbb{P}(\mathcal{H})$ and $\tilde{\mathbb{X}} = \text{Segre}(\mathbb{P}(U_1) \times \cdots \times \mathbb{P}(U_L)) \subset \mathbb{P}(\tilde{\mathcal{H}})$; these are the varieties of decomposable tensors in the two systems. We have $\tilde{\mathbb{X}} = \mathbb{X} \cap \mathbb{P}(\tilde{\mathcal{H}})$.

Case 2: Bosons. Let \mathcal{H}_1 be a finite-dimensional vector space and $\mathcal{H} = S^L(\mathcal{H}_1)$. Let $U \subset \mathcal{H}_1$ be a subspace and $\tilde{\mathcal{H}} = S^L(U)$. We have natural embeddings $\tilde{\mathcal{H}} \subset \mathcal{H}$ and $\mathbb{P}(\tilde{\mathcal{H}}) \subset \mathbb{P}(\mathcal{H})$. The varieties of coherent states are $\mathbb{X} = \text{Veronese}_L(\mathbb{P}(\mathcal{H}_1)) \subset \mathbb{P}(\mathcal{H})$ and $\tilde{\mathbb{X}} = \text{Veronese}_L(\mathbb{P}(U)) \subset \mathbb{P}(\tilde{\mathcal{H}})$. We have $\tilde{\mathbb{X}} = \mathbb{X} \cap \mathbb{P}(\tilde{\mathcal{H}})$.

Case 3: Fermions. Let \mathcal{H}_1 be a finite-dimensional complex vector space and $\mathcal{H} = \bigwedge^L \mathcal{H}_1$. Let $U \subset \mathcal{H}_1$ be a subspace with $\dim U \leq L$ and let $\tilde{\mathcal{H}} = \bigwedge^L U$. We have natural embeddings $\tilde{\mathcal{H}} \subset \mathcal{H}$ and $\mathbb{P}(\tilde{\mathcal{H}}) \subset \mathbb{P}(\mathcal{H})$. The varieties of coherent states are the Plücker embeddings of the Grassmann varieties $\mathbb{X} = \text{Pl}(\text{Gr}(L, \mathcal{H}_1)) \subset \mathbb{P}(\mathcal{H})$ and $\tilde{\mathbb{X}} = \text{Pl}(\text{Gr}(L, U)) \subset \mathbb{P}(\tilde{\mathcal{H}})$. We have $\tilde{\mathbb{X}} = \mathbb{X} \cap \mathbb{P}(\tilde{\mathcal{H}})$.

Lemma 7.3. Suppose we are in one of the three situations described above. Let $[\psi] \in \mathbb{P}(\tilde{\mathcal{H}})$. Then,

$$\text{rk}_{\mathbb{X}}[\psi] = \text{rk}_{\tilde{\mathbb{X}}}[\psi].$$

Proof. Since the proofs in the three cases are completely analogous, we shall only treat the case of bosons. Denote $r = \text{rk}_{\mathbb{X}}[\psi]$ and let

$$\psi = x_1 + \cdots + x_r, \quad [x_l] \in \mathbb{X},$$

be a minimal expression for ψ . We have $x_l = y_l^L$ for some $y_l \in \mathcal{H}_1$. We want to show that $[x_l]$ s may actually be chosen in $\tilde{\mathbb{X}}$, which means that y_l s can be chosen in U .

Let $u_1, \dots, u_m, v_1, \dots, v_{n-m}$ be a basis of \mathcal{H}_1 such that u_1, \dots, u_m is a basis of U . We can write y_l in this basis, say

$$y_l = \sum_{i=1}^m a_l^i u_i + \sum_{j=1}^{n-m} b_l^j v_j.$$

We can use these expressions to write ψ in terms of the monomial basis of $S^L(\mathcal{H}_1)$. Since $\psi \in S^L(U)$, the final expression for ψ will have nonzero coefficients only in front of the monomials in the u_j ; no v_j will be included. But now, observe that one obtains the same result if instead of y_l , one takes

$$z_l = \sum_{i=1}^m a_l^i u_i.$$

So, we have

$$z_1^k + \cdots + z_r^k = y_1^k + \cdots + y_r^k = \psi.$$

Setting $x'_l = z_l^k$, we obtain $[x'_l] \in \tilde{\mathbb{X}}$ and $\psi = x'_1 + \cdots + x'_r$ as desired. \square

7.1.2. Basic case for distinguishable particles: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Here, we construct an explicit approximation by states of rank 2 for the W -state of a system of three qubits. Let v_1, v_2 be a basis of \mathbb{C}^2 . In fact, the state which is LU equivalent to W (of the same rank as W) is obtained as a limit point of the action of some one-parameter subgroup of $GL_2(\mathbb{C})$. The one-parameter subgroup is

$$A(a) = \frac{1}{2} \begin{pmatrix} a + a^{-1} & a - a^{-1} \\ a - a^{-1} & a + a^{-1} \end{pmatrix}, \quad a \in \mathbb{C}^\times,$$

which may also be written as

$$A(a) = g_0 A_1(a) g_0^{-1}, \quad \text{where } A_1(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in GL_2(\mathbb{C}), \quad g_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in SU(2).$$

A direct calculation shows that we have the following convergence in $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$.

$$A(a)^{\otimes 3} [v_1 \otimes v_1 \otimes v_1 + v_2 \otimes v_2 \otimes v_2] \xrightarrow{a \rightarrow 0} g_0^{\otimes 3} [v_1 \otimes v_2 \otimes v_2 + v_2 \otimes v_1 \otimes v_2 + v_2 \otimes v_2 \otimes v_1].$$

So the state $g_0 W$, and hence W , can be approximated by states of rank 2. On the other hand, it can be checked by a direct calculation that $\text{rk}[W] = 3$. Thus, W is an exceptional state. For an argument which uses the orbit structure, see remark 7.2.

7.1.3. Bosons: $S^L(\mathbb{C}^n)$. Here, we show that if $L \geq 3$, then the system $S^L(\mathbb{C}^n)$ of L bosons has exceptional states.

Let v_1, \dots, v_n be a basis of \mathbb{C}^n , so that the monomials of degree L in v_j form a basis of $\mathcal{H} = S^L \mathbb{C}^n$. Let $\mathbb{X} = \text{Ver}_L(\mathbb{P}^{n-1}) \subset \mathbb{P}(\mathcal{H})$. Let

$$\psi = v_1^L + (-1)^{L+1} v_n^L. \quad (7)$$

Consider the one-parameter subgroup of $G = GL_n(\mathbb{C})$ given by

$$A(a) = \begin{pmatrix} \frac{a+a^{-1}}{2} & 0 & \frac{a-a^{-1}}{2} \\ 0 & \mathbb{I}_{n-2} & 0 \\ \frac{a-a^{-1}}{2} & 0 & \frac{a+a^{-1}}{2} \end{pmatrix}, \quad a \in \mathbb{C}^\times.$$

We have $A(a) = g_0 A_1(a) g_0^{-1}$, where

$$A_1(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & \mathbb{I}_{n-2} & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \in SL_n \mathbb{C}, \quad g_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \mathbb{I}_{n-2} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

A simple application of the binomial formula shows that

$$A(a)[\psi] \xrightarrow{a \rightarrow 0} g_0[v_1 v_n^{L-1}] \quad \text{in } \mathbb{P}(\mathcal{H}).$$

Hence,

$$\underline{\text{rk}}[v_1 v_n^{L-1}] = 2.$$

But, according to [12] or [8] (the relevant result is stated here as theorem 5.4), we have

$$\text{rk}[v_1 v_n^{L-1}] = L.$$

Thus,

$$L > 2 \implies \text{rk} \neq \underline{\text{rk}}.$$

Finally, note that $[v_1 v_n^{L-1}]$ can be regarded as a W -type state of L bosons, i.e. a symmetric state in which $L - 1$ out of L bosons are in the highest excited state.

7.1.4. Basic case for fermions: $\bigwedge^3 \mathbb{C}^6$. Let $G = SL_6(\mathbb{C})$, $\mathcal{H} = \bigwedge^3 \mathbb{C}^6$, $\mathbb{X} = \text{Pl}(\text{Gr}_3(\mathbb{C}^6)) \subset \mathbb{P}(\mathcal{H})$. We shall show that $\mathbb{P}(\mathcal{H})$ contains exceptional states with respect to \mathbb{X} .

Let $\mathbb{P} = \mathbb{P}(\mathcal{H})$. We have $\dim \mathbb{X} = 9$ and $\dim \mathbb{P} = 19 = 2 \dim \mathbb{X} + 1$. The ring of invariants is isomorphic to a polynomial ring in one variable, $\mathbb{C}[\mathcal{H}]^G = \mathbb{C}[F]$ and the generator F is a polynomial with $\deg F = 4$ (cf [18], table II). Let $Z \subset \mathbb{P}$ be the quartic hypersurface defined by the vanishing of F . Then, Z is a G -invariant variety and $\mathbb{X} \subset Z \subset \mathbb{P}$.

Let $\mathcal{N}_\alpha = \{\psi \in \mathcal{H} : F(\psi) = \alpha\}$, for $\alpha \neq 0$. Proposition 3.3 in [18] states that \mathcal{N}_α is a single G -orbit. It follows that G acts transitively on $\mathbb{P} \setminus Z$. Hence, the latter set consists exactly of the generic states of rank 2. It remains to understand the orbit structure in Z (cf [21]).

We consider now the states of rank 2 in \mathcal{H} and show that there are two types of such states. Any such state can be written as

$$\psi = v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6,$$

with some $v_j \in \mathcal{H}$. The first possibility is that v_1, \dots, v_6 form a basis of \mathbb{C}^6 . This is indeed the generic situation. If suitable Borel and Cartan subgroups of SL_6 are chosen, the two summands of ψ are, respectively, the highest and lowest weight vectors in \mathcal{H} . The group GL_6 acts transitively on the set of all bases of \mathbb{C}^6 ; the group SL_6 acts transitively on the set of their projective images. Thus, the states of the first type form a single G -orbit $\mathbb{X}'_2 = \mathbb{P} \setminus Z$. The second possibility is to have

$$\dim(\text{span}\{v_1, v_2, v_3\} \cap \text{span}\{v_4, v_5, v_6\}) = 1.$$

If this is the case, by changing the vectors if necessary, we may reduce to the situation where $v_1 = v_4$ and

$$\psi = v_1 \wedge (v_2 \wedge v_3 + v_5 \wedge v_6), \quad \text{with} \quad \text{span}\{v_2, v_3\} \cap \text{span}\{v_5, v_6\} = 0.$$

Since $v_2 \wedge v_3 + v_5 \wedge v_6$ has rank 2 in $\bigwedge^2 \mathbb{C}^6$ (with respect to $\text{Gr}(2, \mathbb{C}^6)$), we deduce that ψ has indeed rank 2 in \mathcal{H} . The state ϕ is not of the first type described above because the action of GL_6 respects linear dependences. On the other hand, it is also clear that GL_6 acts transitively on the set \mathbb{X}_2' of states of this second type, and hence SL_6 acts transitively on the set of their images in \mathbb{P} . Note that if

$$\dim(\text{span}\{v_1, v_2, v_3\} \cap \text{span}\{v_4, v_5, v_6\}) > 1,$$

then $\text{rk}[\psi] = 1$. We can conclude that there are exactly two G -orbits consisting of states of rank 2, namely

$$\mathbb{X}_2 = \mathbb{X}_2' \sqcup \mathbb{X}_2'', \quad \mathbb{X}_2' = \mathbb{P} \setminus Z, \quad \mathbb{X}_2'' = Z \cap \mathbb{X}_2.$$

Now, we verify the presence of states of rank 3. Set

$$\phi = v_{1,1} \wedge v_{1,2} \wedge v_{1,3} + v_{2,1} \wedge v_{2,2} \wedge v_{2,3} + v_{3,1} \wedge v_{3,2} \wedge v_{3,3},$$

with $v_{j,k} \in \mathcal{H}$. Denote $V_j = \text{span}\{v_{j,1}, v_{j,2}, v_{j,3}\}$ for $j = 1, 2, 3$. If $\dim(V_i \cap V_j) > 1$ for some pair of distinct indices, then the corresponding two summands of ϕ can be joined into a single simple tensor, and hence ϕ has rank at most 2. If $\dim(V_i \cap V_j) = 0$ for some pair of distinct indices, say $\dim(V_1 \cap V_2) = 0$, then either $\dim(V_1 \cap V_3) > 1$ or $\dim(V_2 \cap V_3) > 1$, and we are brought to the previous case. Suppose $\dim(V_i \cap V_j) = 1$ for all pairs of distinct indices. Here is an example of such a state:

$$\phi = v_1 \wedge v_2 \wedge v_4 - v_1 \wedge v_3 \wedge v_5 + v_2 \wedge v_3 \wedge v_6,$$

where $\{v_1, \dots, v_6\}$ is a basis of \mathbb{C}^6 . To show that this state has indeed rank 3, we propose the following argument. If ϕ had rank 2, then it must belong either to \mathbb{X}_2' or \mathbb{X}_2'' according to the above construction. It does not belong to \mathbb{X}_2'' because $V_1 + V_2 + V_3 = \mathbb{C}^6$, while the components of any state from \mathbb{X}_2'' span a five-dimensional subspace of \mathbb{C}^6 . For an analogous reason, $[\phi]$ does not belong to \mathbb{X} . To show that ϕ does not belong to \mathbb{X}_2' , we simply observe that ϕ is the limit point of an orbit of a one-parameter subgroup of G , through a point in \mathbb{X}_2' . Take

$$[\psi] = [v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6] \in \mathbb{X}_2'.$$

Consider the one-parameter subgroup of GL_6 given by

$$A(a) = \frac{1}{8} \begin{pmatrix} a + a^{-1} & 0 & 0 & 0 & 0 & a - a^{-1} \\ 0 & a + a^{-1} & 0 & 0 & a - a^{-1} & 0 \\ 0 & 0 & a + a^{-1} & a - a^{-1} & 0 & 0 \\ 0 & 0 & a - a^{-1} & a + a^{-1} & 0 & 0 \\ 0 & a - a^{-1} & 0 & 0 & a + a^{-1} & 0 \\ a - a^{-1} & 0 & 0 & 0 & 0 & a + a^{-1} \end{pmatrix}, \quad a \in \mathbb{C}^\times.$$

It can also be written as $A(a) = g_0 A_1(a) g_0^{-1}$, with the following $A_1(a) \in SL_6 \mathbb{C}$ and $g_0 \in SU_6$:

$$A_1(a) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & a^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & a^{-1} \end{pmatrix}, \quad g_0 = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A direct calculation shows that

$$A(a)[\psi] \xrightarrow{a \rightarrow 0} g_0[v_1 \wedge v_2 \wedge v_4 - v_1 \wedge v_3 \wedge v_5 + v_2 \wedge v_3 \wedge v_6] = g_0[\phi] \quad \text{in } \mathbb{P}(\mathcal{H}).$$

Thus, the state $[g_0\phi]$ is a limit point of the orbit of $A(a)$ through $[\psi]$. We want to show that $[g_0\phi] \notin \mathbb{X}_2'$, which would imply that $[\phi] \notin \mathbb{X}_2'$. To this end, note that both ψ and ϕ are

critical points of the norm of the momentum map which is given by $\|\mu([v])\|^2 = \text{tr}(\mu([v])^2)$. However, the spectra of their reduced density matrices are different. This means that ψ and ϕ are not LU equivalent, i.e. do not belong to the same K -orbit. It is known [22] that any G -orbit can contain at most one K -orbit with critical points of the norm of momentum map [22] (see also [25] for a discussion of $\|\mu([v])\|^2$ properties in the entanglement setting). Hence, ϕ and ψ belong to different G -orbits and $[\phi] \notin \mathbb{X}'_2$.

To summarize, we have $[\phi] \notin \mathbb{X}_2 \cup \mathbb{X}$. We can conclude that

$$\text{rk}[\phi] = 3 \quad \text{and} \quad \underline{\text{rk}}[\phi] = 2,$$

which means that ϕ is an exceptional state. Note that the state ϕ is a fermionic counterpart of the three-qubit W -state.

Remark 7.2. Note that the same reasoning can be mutatis mutandis applied to the three-qubit case to prove that $\text{rk}[W] = 3$. We have exactly one invariant, the so-called hyperdeterminant. Moreover, the action of $SL_2(\mathbb{C})^{\times 3}$ on $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ has a dense open orbit. It can be shown that all states of rank 2 in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ belong to either the open orbit or one of the three orbits of bi-separable states. Clearly, W is not bi-separable. What is left is to show that it does not belong to the open orbit. To this end, once again, note that W and $\psi = v_1 \otimes v_1 \otimes v_1 + v_2 \otimes v_2 \otimes v_2$ are critical points of the norm of the momentum map. However, the spectra of their reduced density matrices are different. This means that W and ϕ are not LU equivalent, i.e. do not belong to the same K -orbit. But any G -orbit can contain at most one K -orbit with critical points of the norm of momentum map [22]. Hence, W and ψ belong to different G -orbits and $\text{rk}[W] = 3$.

We can now finish the proof of theorem 7.2.

Proof (of theorem 7.2). For the three considered cases, we have the following.

Case (i). Let U_1, U_2, U_3 be two-dimensional subspaces of $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$, respectively. Let $U_j \subset \mathcal{H}_j$ be a one-dimensional subspace for $j = 4, \dots, L$. Let $\tilde{\mathcal{H}} = U_1 \otimes \dots \otimes U_L \subset \mathcal{H}$. Then, we fall in the situation described in section 7.1.1 and we shall use the notation introduced therein, in particular $\tilde{\mathbb{X}} = \mathbb{X} \cap \mathbb{P}(\tilde{\mathcal{H}})$. So, by lemma 7.3, for states $[\psi] \in \mathbb{P}(\tilde{\mathcal{H}})$, we have $\text{rk}_{\mathbb{X}}[\psi] = \text{rk}_{\tilde{\mathbb{X}}}[\psi]$. Now, note that $\tilde{\mathcal{H}} = U_1 \otimes U_2 \otimes U_3$ because the one-dimensional factors can be dropped out. We know, from the example given in section 7.1.2 (or from section 2), that the W -state has rank 3 but can be approximated by states of rank 2. Thus, the W -state is exceptional in $\tilde{\mathcal{H}}$ and, consequently, in \mathcal{H} .

Case (ii). Here, we can directly apply the result of the example computed in section 7.1.3.

Case (iii). Let $U \subset \mathcal{H}_1$ be any subspace of dimension $L + 3$. Let $\tilde{\mathcal{H}} = \bigwedge^L U$. Then, $\tilde{\mathcal{H}}$ is naturally included as a subspace of \mathcal{H} and we fall again in the situation treated in section 7.1.1. Put $\tilde{\mathbb{X}} = \mathbb{X} \cap \mathbb{P}(\tilde{\mathcal{H}})$. We can apply lemma 7.3 and so we know that for $[\psi] \in \mathbb{P}(\tilde{\mathcal{H}})$, we have $\text{rk}_{\mathbb{X}}[\psi] = \text{rk}_{\tilde{\mathbb{X}}}[\psi]$. Thus, it is sufficient to find exceptional states in $\tilde{\mathcal{H}}$. Note that, since $\dim U = L + 3$, we have $\bigwedge^L U \cong \bigwedge^3 U$ and the rank functions agree (see remark 4.1). Since $\dim U \geq 6$, we can choose a six-dimensional subspace $Y \subset U$. Put $\tilde{\mathcal{H}} = \bigwedge^3 Y$. Then, $\tilde{\mathcal{H}}$ is a subspace of $\bigwedge^3 U$ to which we can apply once again the reduction procedure from section 7.1.1. It is thus sufficient to find exceptional states in $\bigwedge^3 \mathbb{C}^6$. This is done in section 7.1.4. \square

8. The exceptional states for systems with the known G -orbit structure

So far, we have established the existence of the exceptional states stemming from the second secant variety, σ_2 . In this section, we give a list of all exceptional states which appear for

distinguishable particles when: (a) the number of G -orbits in $\mathbb{P}(\mathcal{H})$ is finite and (b) four-qubit system, where the number of G orbits is infinite but their structure is known. The review is based on [4, 7].

8.1. $\mathbb{P}(\mathcal{H})$ with a finite number of G -orbits

In this subsection, we consider all systems of distinguishable particles for which the number of G -orbits in $\mathbb{P}(\mathcal{H})$ is finite. Note that since for $\mathcal{H} = (\mathbb{C}^2)^{\otimes 4}$ the number of G -orbits in $\mathbb{P}(\mathcal{H})$ is already infinite [4], we can restrict to three-partite systems. The requirement of finite number of G -orbits reduces our considerations to two possibilities [23]:

- (i) $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$, where $N \geq 2$, $G = SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_N(\mathbb{C})$
- (ii) $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^N$, where $N \geq 3$, $G = SL_2(\mathbb{C}) \times SL_3(\mathbb{C}) \times SL_N(\mathbb{C})$.

Case 1. As we already know, when $N = 2$, there are six G -orbits. A complete description of the orbit structure for $N > 2$ can be found in [7] (see table 1). Interestingly, the number of orbits stabilizes for $N \geq 4$. More precisely, when $N = 3$, there are two additional G -orbits compared with $N = 2$, i.e. altogether there are eight G -orbits. Starting from $N = 4$, the number of G -orbits is 9 and does not change with N . The rank and border rank of states belonging to each G -orbit have been calculated and can be found in [7, table 1]. There is only one G -orbit containing exceptional states: the G -orbit through $W = |100\rangle + |010\rangle + |001\rangle$. We have $\text{rk}[W] = 3$ and $\underline{\text{rk}}W = 2$.

Case 2. For $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^N$, the orbit structure is much richer. The number of orbits of $G = SL_2(\mathbb{C}) \times SL_3(\mathbb{C}) \times SL_N(\mathbb{C})$ stabilizes for $N \geq 6$. More precisely,

- $N = 3$: 18 G -orbits
- $N = 4$: 23 G -orbits
- $N = 5$: 25 G -orbits
- $N \geq 6$: 26 G -orbits

The complete list of these orbits together with their dimensions, exemplary states belonging to each one, ranks and border ranks can be found in [7, tables 2–4]. Here, we list the exceptional states, their rank, border rank and the smallest N for which they appear

- (i) $\Psi_1 = |100\rangle + |010\rangle + |001\rangle$, $\text{rk}\Psi_1 = 3$, $\underline{\text{rk}}\Psi_1 = 2$, $N = 2$,
- (ii) $\Psi_2 = |0\rangle \otimes (|00\rangle + |11\rangle) + |1\rangle \otimes (|01\rangle + |22\rangle)$, $\text{rk}\Psi_2 = 4$, $\underline{\text{rk}}\Psi_2 = 3$, $N = 3$,
- (iii) $\Psi_3 = |0\rangle \otimes (|00\rangle + |11\rangle + |22\rangle) + |1\rangle \otimes (|01\rangle + |12\rangle)$, $\text{rk}\Psi_3 = 4$, $\underline{\text{rk}}\Psi_3 = 3$, $N = 3$,
- (iv) $\Psi_4 = |0\rangle \otimes (|00\rangle + |11\rangle + |22\rangle) + |101\rangle$, $\text{rk}\Psi_4 = 4$, $\underline{\text{rk}}\Psi_4 = 3$, $N = 3$,
- (v) $\Psi_5 = |0\rangle \otimes (|00\rangle + |12\rangle) + |1\rangle \otimes (|01\rangle + |22\rangle)$, $\text{rk}\Psi_5 = 4$, $\underline{\text{rk}}\Psi_5 = 3$, $N = 3$,
- (vi) $\Psi_6 = |0\rangle \otimes (|00\rangle + |12\rangle + |23\rangle) + |1\rangle \otimes (|01\rangle + |13\rangle)$, $\text{rk}\Psi_6 = 5$, $\underline{\text{rk}}\Psi_6 = 4$, $N = 4$.

Interestingly, these states correspond to nontrivial entanglement classes found recently by means of the so-called sub-Schmidt decomposition by the authors of [13].

8.2. Infinite number of orbits—four qubits

The number of $G = SL_2(\mathbb{C})^{\times 4}$ -orbits in $\mathbb{P}(\mathcal{H})$ where $\mathcal{H} = (\mathbb{C}^2)^{\otimes 4}$ is infinite. Nevertheless, the orbit structure is explicitly known [4, 27]. Note that, according to theorem 5.2, the minimal number of simple tensors in $(\mathbb{C}^2)^{\otimes 4}$ necessary to express a generic tensor as a linear combination is the expected number $r_g = \lceil \frac{2^4}{4+1} \rceil = 4$. It was proven by Brylinski [3]

that the maximal tensor rank of a four-qubit state is 4. Therefore, tensors of rank 4 cannot approximate tensors of higher rank and the border rank of any exceptional $[\psi] \in \mathbb{P}(\mathcal{H})$ can be at most 3. Consequently, the exceptional states can belong only to σ_2 and σ_3 , i.e. they arise from the closure of states of ranks 2 and 3. These states have been completely determined in [4].

States of rank 2 and σ_2 . By [4, proposition 5.1], the states of rank 2 belong to three G -orbits, with representatives:

- (i) $\Psi_1 = [|0000\rangle + |1111\rangle]$
- (ii) $\Psi_2 = [|0\rangle \otimes (|000\rangle + |111\rangle)]$
- (iii) $\Psi_3 = [|00\rangle \otimes (|00\rangle + |11\rangle)]$.

Therefore, the closures of $G.[\Psi_{1,2,3}]$ are the only source of exceptional states belonging σ_2 . It is easy to see that $\overline{G.[\Psi_3]} \setminus G.[\Psi_3]$ contains only separable states. In $\overline{G.[\Psi_2]}$, we find the state $\phi_2 = |0\rangle \otimes (|001\rangle + |010\rangle + |100\rangle)$ which has rank 3 and border rank 2 and hence is exceptional. Note that this state is a tensor product of $|0\rangle$ and a three-qubit W -state. The last exceptional state-type stems for $\overline{G.[\Psi_1]}$ and is a four-qubit W -state, i.e. $[|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle]$.

States of rank 3 and σ_3 . The second and last source of exceptional states is σ_3 . It was shown by Brylinski [3] that σ_3 is an irreducible algebraic variety given as the zero locus of two out of the four generating invariant polynomials of the G -action on \mathcal{H} (these polynomials are denoted by L and M in [4]). Since the G -orbits in \mathcal{H} are known explicitly [4], one can single out those on which the above-mentioned polynomials vanish. If in addition one knows representatives of the G -orbits of ranks 2 and 3, one can determine the exceptional states stemming from σ_3 . Following this reasoning, the authors of [4] found that the closure of the set of states of rank at most 3 contains only one G -orbit of states of rank 4. This is G -orbit of the four-qubit W -state $[|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle]$.

The conclusion is that, for four qubits, the only exceptional states are G -orbits through $|0\rangle \otimes (|001\rangle + |010\rangle + |100\rangle)$ and $[|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle]$ which are three- and four-qubit W -states.

9. Summary

We have established a connection between spherical actions of a reductive groups and the border rank of a state—a typical notion in secant-variety theory. More precisely, we showed that sphericity implies that states of a given rank cannot be approximated by states of a lower rank, i.e. there are no exceptional states. For three, important from a quantum entanglement perspective, cases of distinguishable, fermionic and bosonic particles, we also show that non-sphericity implies the existence of the exceptional states. We showed that the corresponding exceptional states belong to non-bipartite entanglement classes of W -type. Finally, we concluded that the existence of the exceptional states is a state-type obstruction for deciding the LU equivalence of states by means of the one-particle-reduced density matrices.

A desired result would be a classification of exceptional states for the considered three types of systems, that is, L bosons, fermions and distinguishable particles. Note that even in the case of L qubits, such a classification is not known. All examples of exceptional states in L -qubit systems, known to us, are of W -type. It would be interesting to find out if there are exceptional states of different types. In order to find new examples, one should look at the system of at least five qubits. It is because $\text{rk} W = L$ and the generic rank $\lceil \frac{2^L}{L+1} \rceil > L$, when

$L \geq 5$. For example when $L = 6$, the generic rank is 10 and the rank of W is 6. It would be very surprising if the sequences of tensors of ranks 6, 7, 8, 9 or 10 do not give tensors of higher rank. In our opinion the fact that for four qubits the maximal rank and generic rank overlap, which greatly simplifies the problem, might be a specific low-dimensional phenomenon. Note also that all exceptional states found in the considered examples are unstable states in the sense of geometric invariant theory (GIT). This in turn means that they cannot be taken by SLOCC operations (G -action) to a state with maximally mixed reduced one-particle density matrices. The mutual relationship between classification of states with respect to rank or border rank and with respect to GIT stability seems to be a very interesting problem from both physics and mathematics points of view which we intend to follow.

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